IMPLICATIONS OF PASSIVITY ON POWER DIVISION

Dimitrios Antsos

Jet Propulsion Laboratory
California Institute of Technology
Mail Stop T-1 701
4800 Oak Grove Drive
Pasadena, CA 91109

Tel: (818) 354-6343
FAX: (81 8) 393-0050

1. Abstract

Passive, multi-port power dividers are often used in array-antenna applications. Low loss is an important desired property of these devices. Fundamental limits imposed by passivity and geometrical considerations on the power-transfer performance of these devices are analytically derived and calculated in this paper, using 3-port networks as examples.

2. introduction

This paper is relevant to current research efforts at JPL and in the industry to fabricate multi-way power dividers and combiners for array-antenna applications. The unit-cell of most of these dividers is a 3-port or a 4-port network with one port terminated (e.g., branchlike, ratrace, Wilkinson and Gysel power dividers). As the size of array-antennas increases, so do the required "layers" of power division (as the base-two logarithm of the size). Therefore, the insertion loss of these unit-cell devices, which are cascaded in layers, becomes an important concern and design parameter, The examples of unit-cell devices mentioned above all provide isolated output ports which can be matched to the input and output port normalizing characteristic impedance (usually 50 Ohms). Fundamental limitations imposed by passivity on these 3-port devices are quantified in this paper and a numerical example of what these limitations entail for the case of a generalized 3-

port network is furnished. In other examples of "corporate" type multi-port power dividers without isolation, the output ports are not matched to 50 Ohms. These work well with passive arrays, where the antennas have input impedances close to 50 Ohms, but are not suitable for driving the amplifiers of active arrays, since amplifiers are sensitive to the match at their input. These corporate dividers, which are not 3-ports, are not considered in this paper.

3. An Analytical Statement of Passivity

A network is considered passive when the power incident onto it is greater than or equal to the power reflected from it, for all possible excitations. A network described by the 'a' and 'b' wave parameters is shown in figure 1 below (where a 3-port network is used for depiction purposes). The a-waves are the incident waves and the b-waves are the reflected waves, at each port. These are normalized so that, for example,

$$\frac{1}{2}|a_1|^2 \qquad (1)$$

corresponds to the power carried by the wave incident at port 1. It is easy to see that the total power incident to this network from all ports is

$$P_{inc} = \frac{1}{2}\sum_{i=1}^{N} |a_i|^2 , \qquad (2)$$

where N is the total number of ports of the network.

Similarly, the total power scattered by the network is

$$P_{scat} = \frac{1}{2}\sum_{i=1}^{N} |b_i|^2 . \qquad (3)$$

An alternate way to express equations (2) and (3) in matrix notation is

$$P_{inc} = \frac{1}{2}\mathbf{a}^{+}\mathbf{a} \tag{4}$$

$$P_{scat} = \frac{1}{2}\mathbf{b}^{+}\mathbf{b} \tag{5}$$

where $\mathbf{a}=(a_1,a_2,\ldots,a_N)^T$ and $\mathbf{b}=(b_1,b_2,\ldots,b_N)^T$ and the '(dagger" notation is used to denote conjugate-transpose. By the definition of the S-matrix we have

$$\mathbf{b} = \mathbf{S}\mathbf{a} \quad , \tag{6}$$

where S is an N-by-N matrix that characterizes the response of the network.

Combining the last 3 equations gives the total dissipated power as

$$P_{dis} = \text{Pine} - P_{scat} = \frac{1}{2}(\mathbf{a}^{+}\mathbf{a} - \mathbf{a}^{+}\mathbf{S}^{+}\mathbf{S}\mathbf{a}) = \frac{1}{2}\mathbf{a}^{+}(\mathbf{I} - \mathbf{S}^{+}\mathbf{S})\mathbf{a} = \frac{1}{2}\mathbf{a}^{+}\mathbf{Q}\mathbf{a} \tag{7}$$

where 1 have defined the dissipation matrix, Q, of the network. For any passive network it must be

$$P_{dis} \geq 0, \ \forall \mathbf{a} \quad . \tag{8}$$

Equations (7) and (S) imply that the matrix Q must be non-negative real (i.e., the quadratic form $\mathbf{a}^{+}\mathbf{Q}\mathbf{a}$ must be a non-negative real number for all **a**). Let us examine the implications of this on Q. To form our conclusions we will use the following theorems from matrix theory:

<u>Theorem **1**</u>

Every hermitian matrix has real eigenvalues.

<u>Theorem</u> <u>2</u>

A hermitian matrix has non-negative eigenvalues if and only if it is positive semi-definite.

Theorem 3

For every hermitian matrix there exists a complete set of 01 thonormal eigenvectors.

Clearly, $\mathbf{S}^{+}$ S is a hermitian and positive semi-definite matrix (proof: (S"' S)$^{+}$ = $S^{+}(\mathbf{S}^{+})^{+}$ = $S^{+}S$ and $\mathbf{a}^{+}\mathbf{S}^{+}\mathbf{S})\mathbf{a} = (\mathbf{a}^{+}\mathbf{S}^{+})(\mathbf{Sa}) = (\mathbf{Sa})^{+}(\mathbf{Sa}) = \mathbf{b}^{+}\mathbf{b} \geq \mathbf{0}\, \forall \mathbf{a}$, since the last expression is the square of the norm of the vector **b** and therefore non-negative). Hence, by Theorems 1 and 2, the eigenvalues of S' S are all real and non-negative, i.e.,

$$\lambda_i^{\mathbf{S^{+}S}} \geq 0, \ \forall i \quad . \tag{9}$$

The matrix Q is also hermitian (proof: $\mathbf{Q}^{+} = (1 - \mathbf{S}^{+}\mathbf{S})^{+} = \mathbf{1} + \text{--} (\mathbf{S}^{+}\mathbf{S})^{+} = \mathbf{I} - \mathbf{S}^{+}\mathbf{S} = \mathbf{Q}$) and its eigenvalues are given by

$$\lambda_i^{\mathbf{Q}} = 1 - \lambda_i^{\mathbf{S^{+}S}} \ \ \forall i \quad . \tag{1o}$$

Proof:

Assume $\lambda_i^{\mathbf{Q}}$ is an eigenvalue of Q.

Hence we have

$$\det\left(\mathbf{Q} - \lambda_i^{\mathbf{Q}}\mathbf{1}\right) = \det\left(\mathbf{I} - \mathbf{S}^{+}\mathbf{S} - \lambda_i^{\mathbf{Q}}\mathbf{I}\right) = -\det\left[\mathbf{S}^{+}\mathbf{S} - \left(1 - \lambda_i^{\mathbf{Q}}\right)\mathbf{I}\right] = \mathrm{O}$$

i.e., $1 - \lambda_i^{\mathbf{Q}}$ is an eigenvalue of S'S, QED.

Using (9) and (1 O) we conclude that

$$\lambda_i^{\mathbf{Q}} \leq 1, \ \forall i \quad . \tag{11}$$

Hence, using Theorems 1 and 2, the hermiticity of Q and equation (11) we can state the necessary and sufficient condition of passivit y (equation (7)) in the following theorem.

Theorem

$$\text{Passivity} \ \Leftrightarrow \mathrm{O} \leq \lambda_i^{\mathbf{Q}} \leq 1\, \forall i \ . \tag{12}$$

(Actually the right part of the right-hand-side of equation (12) is guaranteed, as has been proven in equation (1 1), but it is a good check on any calculations).

Equation (12) states, in words, that the eigenvalues of **Q** (which are real since it is hermitian) must be between O and 1 (greater than or equal to zero because of passivity and less than or equal to one because of the positive-semi-definiteness of $\mathbf{S}^{+}\mathbf{S}$).

An alternate proof of the above theorem, which gives more insight into he physical significance of the eigenvalues and eigenvectors of Q is as follows:

Let us assume that we excite the network of figure 1 with an incident wave $\mathbf{a}_*$, an eigenvector of matrix Q, of power P_{inc}, that corresponds to an eigenvalue λ_*^{Q}. By definition, $\mathbf{a}_*$ must obey the following two equations:

$$\frac{1}{2}\mathbf{a}_*^{+}\mathbf{a}_* = P_{inc} \tag{13}$$

and

$$\mathbf{Q}\mathbf{a}_* = \lambda_*^{Q}\mathbf{a}_* \ . \tag{14}$$

Substituting (13) and (14) into (7) we obtain

$$P_{dis} = \frac{1}{2}\mathbf{a}_*^{+}\mathbf{Q}\mathbf{a}_* = \$\% = \lambda_*^{Q}\left(\frac{1}{2}\mathbf{a}_*^{+}\mathbf{a}_*\right) = \lambda_*^{Q}P_{inc} \tag{15}$$

By the definition of dissipated power we have

$$0 < P_{dis} \leq P_{inc} \ . \tag{16}$$

Substituting (15) into (16) we obtain the desired result, as expressed in equation (12).

Equation (15) provides a good physical interpretation of the significance of the eigenvalues of mat rix Q: *An eigenvalue of Q represents the fraction of the incident power that is dissipated in the network, when the latter is excited by the eigenvector corresponding to that eigenvalue.*

Equation (15) also tells us how to minimize the power dissipated in the network: *Excite the network with an incident wave which is the eigenvector of Q that corresponds 10 its minimum eigenvalue*. In fact, if Q has a zero eigenvalue, i.e., it is singular, it is possible to excite the network in a way that no power is dissipated (with the eigenvector that corresponds to the zero eigenvalue).

4. implications **of Passivity. An Example.**

4.1 The Circuit

A 3-port network is chosen to demonstrate the concepts described above. A representative circuit is a branchlike power divider with its isolated port terminated in a 50 Ohm load (figure *2).* One of the three remaining ports (port 1) is the input and the other two (ports 2 and 3) the outputs. In an idealized model, where all the lines are exactly one quarter of a wavelength long at the design frequency and lossless (no ohmic losses), the isolation between ports 2 and 3 can be analytically shown to be infinite (i.e., S23=0). However, the realities of building the circuit on a substrate are different. On the actual mask, used to fabricate the circuit, one can see that, at the point where the mutually perpendicular quarter-wave lines join, there is a 'T'-junction. This T-junction has dimensions and cannot be considered a "lumped" element. This implies that the microwave current distributes throughout the T-junction; it does not travel a unique path. There is an infinite number of linear paths along which the electrical length of the current is 90 degrees. There is also an infinite number of linear paths along which the electrical length is slightly different from 90 degrees. In the frequency response of the circuit, this has the effect of "broadening" and "shallowing" the infinite well that the magnitude plot of S23 ideally exhibits at the design frequency. This effect is unrelated to ohmic losses (i.e., it also occurs in the ideal, perfectly conducting circuit), It implies that, in an actual circuit of this type, the magnitude of S23 can be very small, but not zero, at the design frequency. This rationale, which is geometry dependent but not material property dependent, allows us to set a lower bound on S23. This bound, together with the passivity constraints on the Q matrix, yield constraints on the insertion loss of this circuit which are not material dependent. Regardless of how lossless the guiding conductor and substrate are, the constraints derived below apply nevertheless.

4.2 **The Calculations**

The software package Matlab, by Mathworks Inc., is used for the analysis. Matlab is preferred because it is optimized for matrix computations and quite accurate in eigenvalue problems. Three different cases are analyzed. The methodology is the same in all three cases:

1. Assume a form for the S-matrix of the network.
2. Compute the eigenvalues of the Q-matrix as a funct ion of insertion loss and isolation.
3. Plot, in the two-dimensional space defined by the insertion loss and the isolation, the curve demarcating the region where the 3-port is passive (i.e., realizable with passive components) from the region where it is not. (i.e., plot the locus of points that lie where the minimum eigenvalue of Q crosses from positive to negative values).

An alternative method to determine the physically achievable region, where Q is passive, is to find the locus of points, in the isolation-insertion loss space, that make the matrix Q singular. This approach should, however, be taken with caution to avoid trivial roots of the characteristic polynomial of Q.

Case 1. Perfectly matched 3-dB power divider w/ finite isolation and insertion loss.

The (symmetric) S-matrix is assumed to have the form

$$
s = \begin{bmatrix} 0 & \frac{\alpha}{\sqrt{2}} & \frac{\alpha}{\sqrt{2}} \\ \frac{\alpha}{\sqrt{2}} & 0 & x \\ \frac{\alpha}{\sqrt{2}} & x & 0 \end{bmatrix},
$$

where a *is* the insertion loss and x the isolation between the output ports of the divider.

The diagonal elements of the matrix (S11, S22 and S33) are assumed zero, (i.e., the device is perfectly matched). S21 and S31 would ideally have the value $1/\sqrt{2}$ (3 dB power divider) and a is the loss in excess of the ideal division loss (insertion loss). The result of the analysis is shown in figure 3. The horizontal axis is the negative of the isolation (i.e., S32, since isolation is defined positive) in dB. The vertical axis is the negative of the insertion loss. On the locus of points of the plotted curve, the minimum eigenvalue of matrix Q is exactly zero. Therefore, the minimum possible dissipated power, for this network, may be achieved by exciting the network with an eigenvector of Q that corresponds to this zero eigenvalue. The equation of the zero-eigenvalue locus of points that make zero dissipation possible, for this form of S-matrix, is

$$
\alpha = \sqrt{1 - x} \ . \tag{17}
$$

The corresponding normalized, unit-power eigenvector is

$$
\begin{pmatrix} \sqrt{\frac{2 - 2x}{2 - x}} \\ \frac{1}{\sqrt{2 - x}} \\ \frac{1}{\sqrt{2 - x}} \end{pmatrix} . \tag{18}
$$

The “achievable” region for a passive circuit is below the curve. A typical value of isolation to be exhibited by the type of 3-port power dividers mentioned above is 20 dB (seldom more than 30 dB). Here 1 treat the isolation as the independent variable and read what the achievable insertion loss is for each 3-port. The point of comparison will be 18 dB isolation. In this case, for 18 dB isolation, the minimum achievable insertion loss is 0.59 dB. As the isolation tends to infinity (i.e.,

$x=0$ in the S-matrix) the minimum achievable insertion loss tends to O dB, as is to be expected for an ideal circuit.

Figure 4 is a plot of the power in each of the three components of the unit-power, zero-dissipation eigenvector (equation (16)) versus the isolation. The usual excitation in microwave circuits is a wave incident to port 1, the input port. Hence, the closer the minimum-loss eigenvector (equation (16)) is to the vector $(\sqrt{2}\ 0\ 0)$, the closer we can come to realizing the zero-dissipation condition.

Case 2. Imperfectly matched 3-dB power divider.

The S-matrix is assumed to have the form

$$s = \begin{bmatrix} 0.1 & \frac{\alpha}{\sqrt{2}} & \frac{\alpha}{\sqrt{2}} \\ \frac{\alpha}{\sqrt{2}} & 0.1 & x \\ \frac{\alpha}{\sqrt{2}} & x & 0.1 \end{bmatrix},$$

for sub-case i. and

$$s = \begin{bmatrix} 0.1 & \frac{\alpha}{\sqrt{2}}\angle -90 & \frac{\alpha}{\sqrt{2}}\angle -90 \\ \frac{\alpha}{\sqrt{2}}\angle -90 & 0.1 & x\angle -180 \\ \frac{\alpha}{\sqrt{2}}\angle -90 & x\angle -180 & 0.1 \end{bmatrix},$$

for sub-case ii (ideal 0°/180° ratrace power divider).

In this case, an equal-phase 20 dB return loss is assumed on all ports, Figure 5 shows the results of the analysis for these matrices. The lower and upper curves show the analysis results for sub-cases i. and ii. respectively.

Sub-case i.

The minimum achievable insertion loss at 18 dB isolation increases to 1.56 dB. However, this case is too restrictive as all the components of the S-matrix are "forced" to be in phase. It is instructive however to note that the trade-off between insertion loss and isolation also depends on the required phase through the circuit. In this case the minimum insertion loss tends to 0.92 dB.

The equation of the zero-eigenvalue locus of points plotted for this sub-case is

$$\alpha = \frac{1}{10}\sqrt{81-90x} \tag{19}$$

and the unit-power, zero-dissipation eigenvector is

$$\begin{bmatrix} \sqrt{\dfrac{18-20x}{18-10x}} \\ \dfrac{3}{\sqrt{18-10x}} \\ \dfrac{3}{\sqrt{18-10x}} \end{bmatrix} \tag{20}$$

Figure **6 is** a plot of the power in each of the three components of the unit-power, zero-dissipation eigenvector (equation (20)) versus the isolation.

Sub-case ii.

The phases of the elements of the S-matrix are set to the values of an ideal 0°/180° ratrace power divider. The minimum achievable insertion loss at 18 dB isolation is 0.56 dB. As the isolation tends to infinity (i.e., x= *0* in the S-matrix) the minimum insertion loss tends to 0.04 dB (which corresponds to the expected loss, in the ideal case, due to the reflected power $|S_{11}|^2$).

The equation of the zero-eigenvalue locus of points plotted for this sub-case is

$$\alpha = \frac{1}{10}\sqrt{99-90x} \tag{21}$$

and the unit-power, zero-dissipation eigenvector is

$$\begin{pmatrix} -j\sqrt{\dfrac{11-10x}{10-5x}} \\ \dfrac{3}{\sqrt{20-10x}} \\ \dfrac{3}{\sqrt{20-10x}} \end{pmatrix} . \tag{22}$$

Figure 7 is a plot of the power in each of the three components of the unit-power, zero-dissipation eigenvector (equation (22)) versus the isolation.

<u>Case 3.2:3 power divider with phases from a measured Wilkinson type power divider.</u>

To relax the constraint that all the elements of the S-matrix are in phase, the measured phases of all the elements of the S-matrix of an actual 2:3 Wilkinson power divider, centered at 30 GHz, are used. The assumed S-mat rix is

$$\mathbf{s} = \begin{bmatrix} 0.1\angle-17 & \alpha\sqrt{\frac{3}{5}}\angle-55 & \alpha\sqrt{\frac{2}{5}}\angle-55 \\ \alpha\sqrt{\frac{3}{5}}\angle-55 & 0.1\angle-117 & x\angle-6 \\ \alpha\sqrt{\frac{2}{5}}\angle-55 & x\angle-6 & 0.1\angle-83 \end{bmatrix},$$

with the usual definitions of x and a. The results of the analysis arc shown in figure 8. The minimum insertion loss at 18 dB isolation is 1.1 dB. The minimum insertion loss tends to 0.88 dB, as the isolation tends to infinity. The above results are not sensitive to "adding line lengths" at the input and output ports.

The equation of the zero-eigenvalue locus of points plotted for this sub-case is

$$\alpha = \frac{\sqrt{9{,}99.10^{4} - 813x - 1.00.10^{5}x^{2} - 513x^{3} - \sqrt{\begin{array}{l}3.73.10^{8} - 1.47\cdot 10^{8}x + 8.56\cdot 10^{9}x^{2} + 2.95\cdot 10^{7}x^{3} \\ -1.84\cdot 10^{10}x^{4} + 1.20\cdot 10^{8}x^{5} + 9.50\cdot 10^{9}x^{6}\end{array}}}}{\sqrt{99080 - 164.1x - 96000x^{2}}} \quad (23)$$

The algebraic expression for the zero-dissipation eigenvector is too complicated and is not included. Figure 9 is a plot of the power in each of the three components of the unit-power, zero-dissipation eigenvector versus the isolation.

S. Conclusions

This paper illustrates that there exist fundamental considerations, above and beyond dielectric loss tangent and conductivity of metalization, that limit the power-transfer performance of passive multi-port networks. Bounds or restrictions unrelated to ohmic losses which can be levied on the S-parameters of a network, may imply additional restrictions, imposed by passivity, on the power-transfer performance of the network, In the examples above, insertion losses, imposed by passivity, are calculated for matched 3-port power dividers, These losses are of a fundamental and inevitable nature. If they are unbearable to the design engineer, other system-design alternatives have to be considered. In particular the above results show that a matched power divider without isolation is very lossy (see figs. 2-4 @ 5 dB isolation). The contra-positive of the above statement, a useful corollary, is that if a power divider without isolation has low insertion loss it cannot have a low return loss on all its ports.

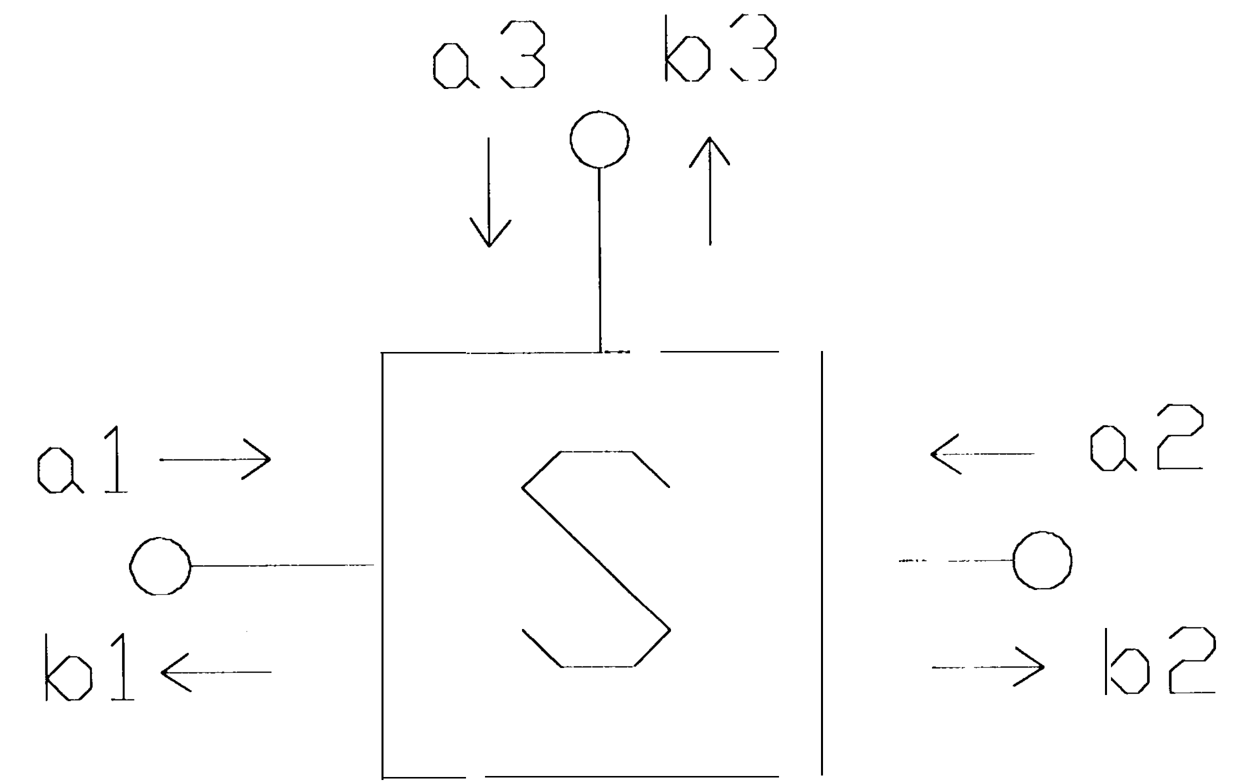


Figure 1 A 3-port network.

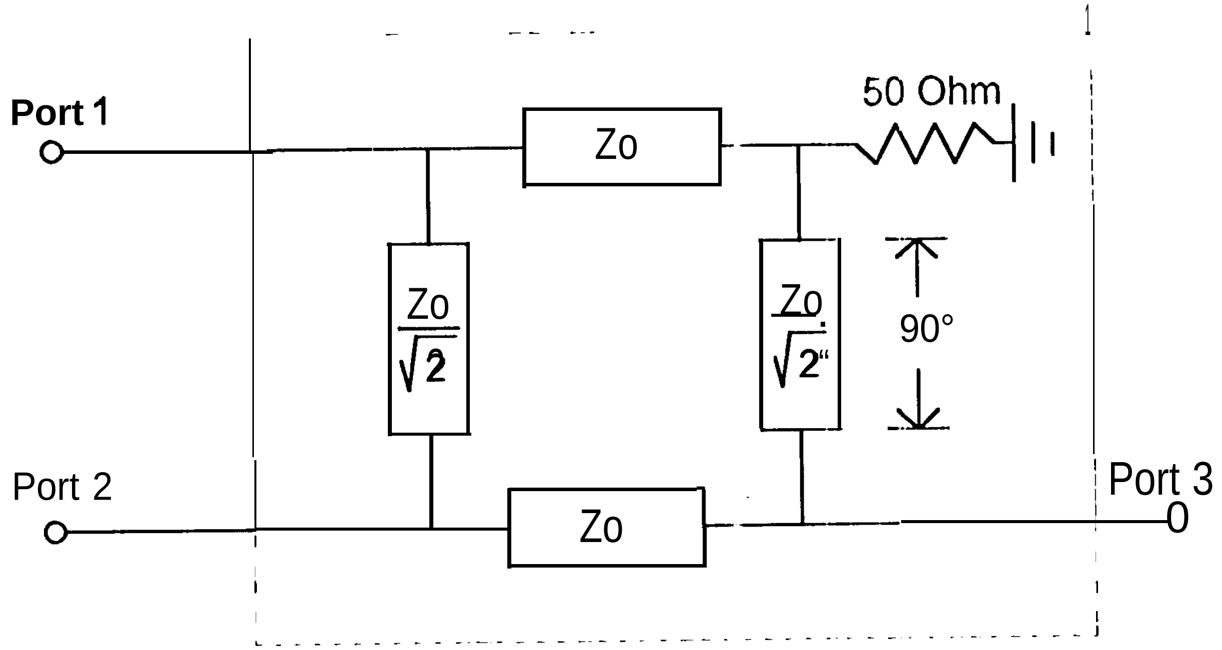


Figure 2 A branch-line power divider.

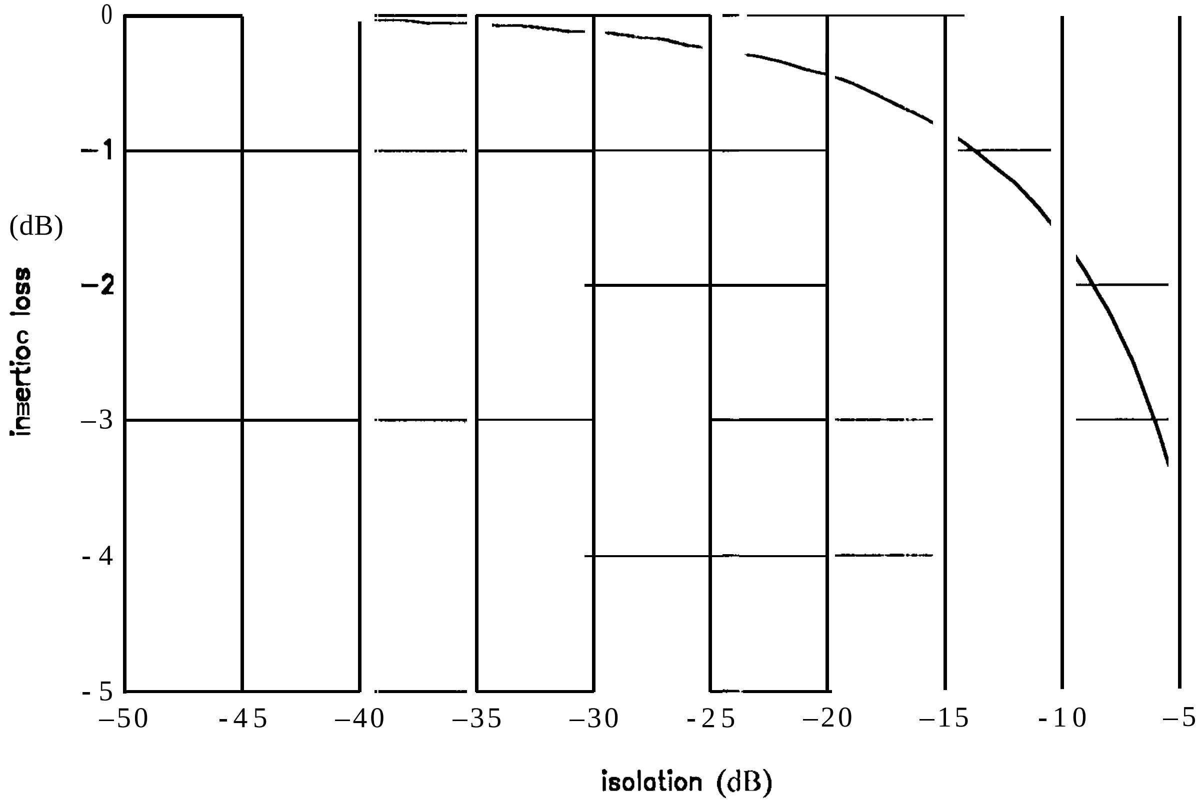


Figure 3 Trade-off between insertion loss & isolation.

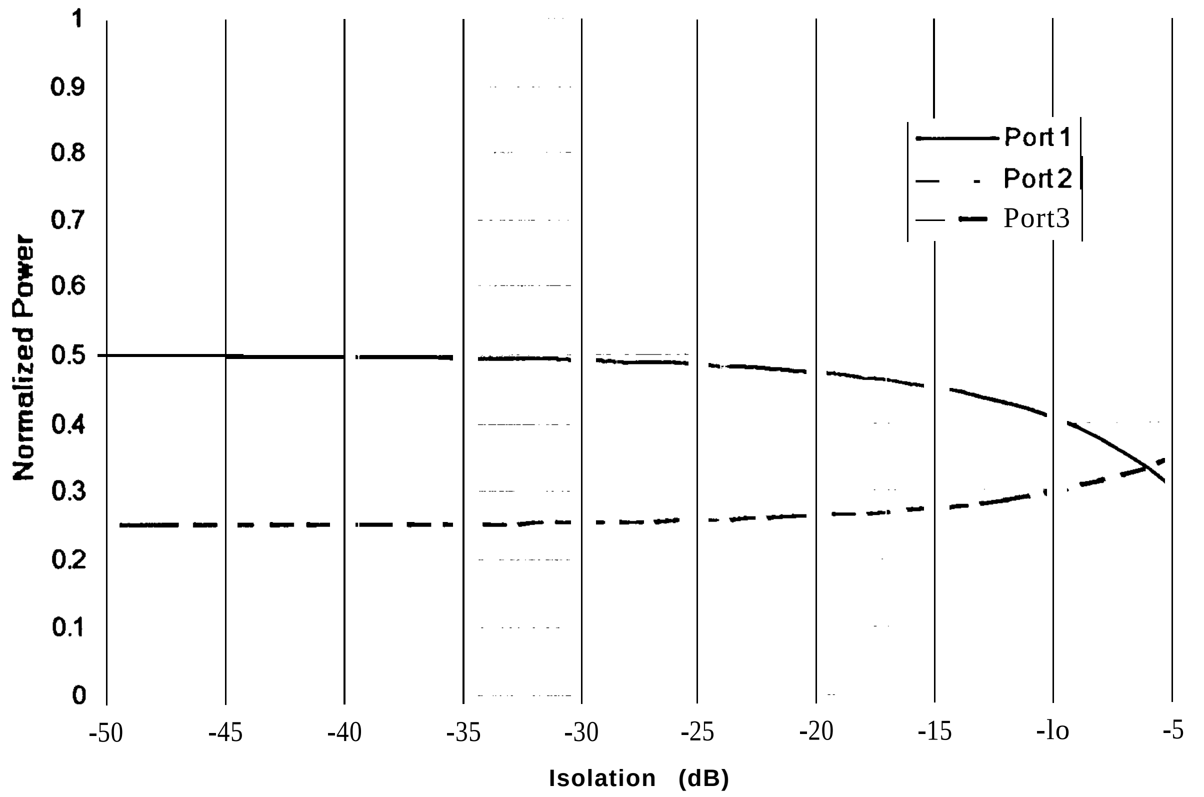


Figure 4 The power, in each of the pork, for zero dissipation.

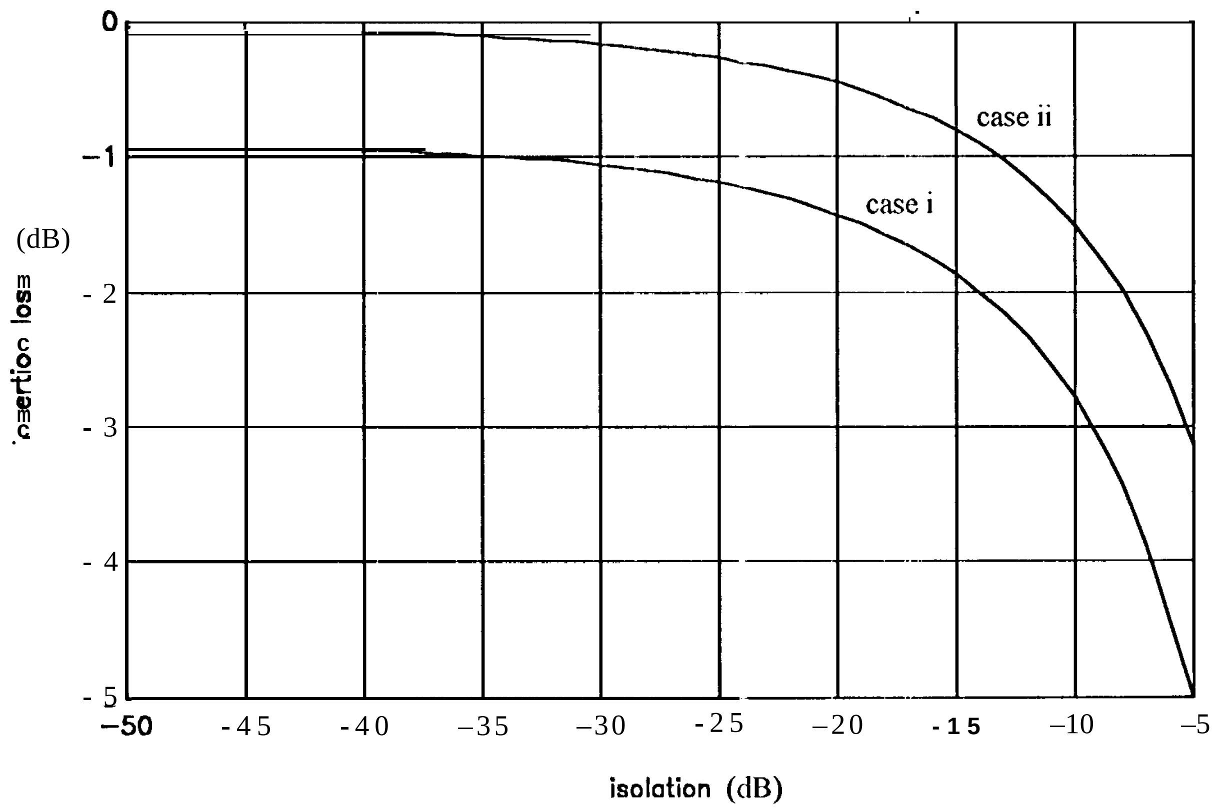
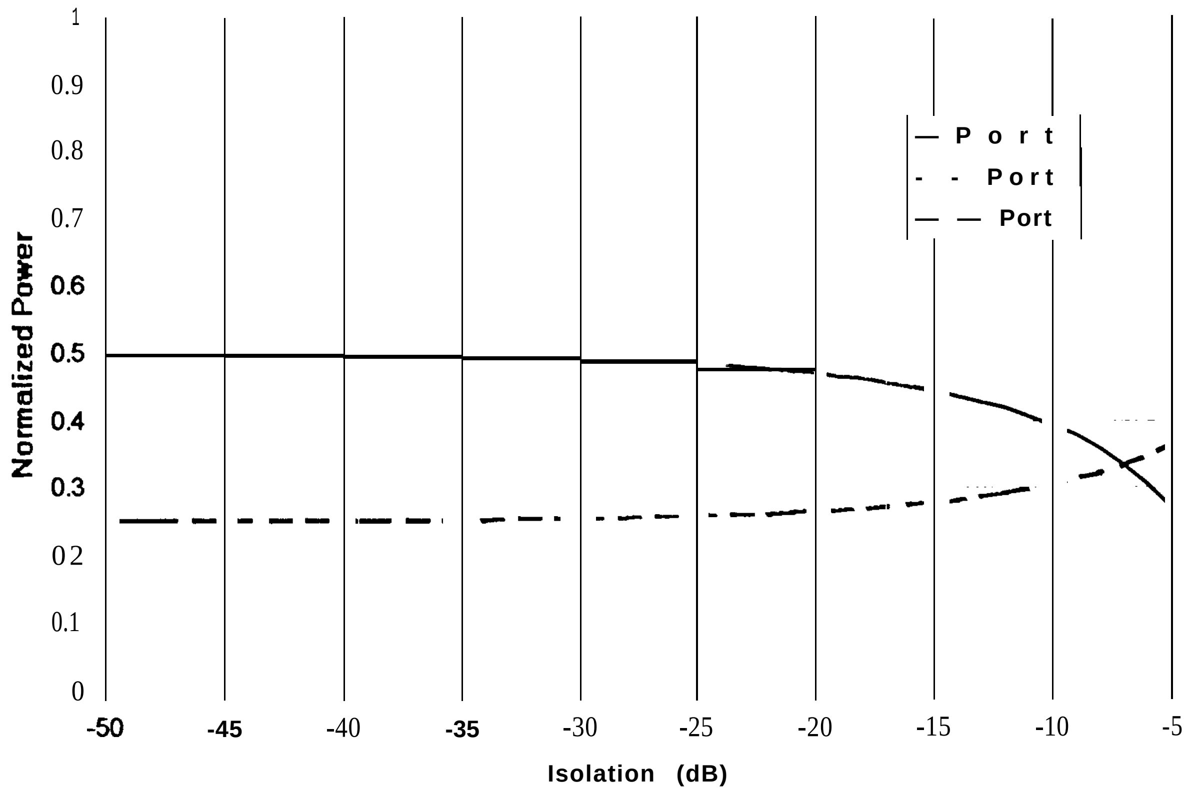


Figure S Trade-ofibctwccn **inscrtion** 1oss & isolation.



ʾigure 6 The power, in each of the ports, for zero dissipation.

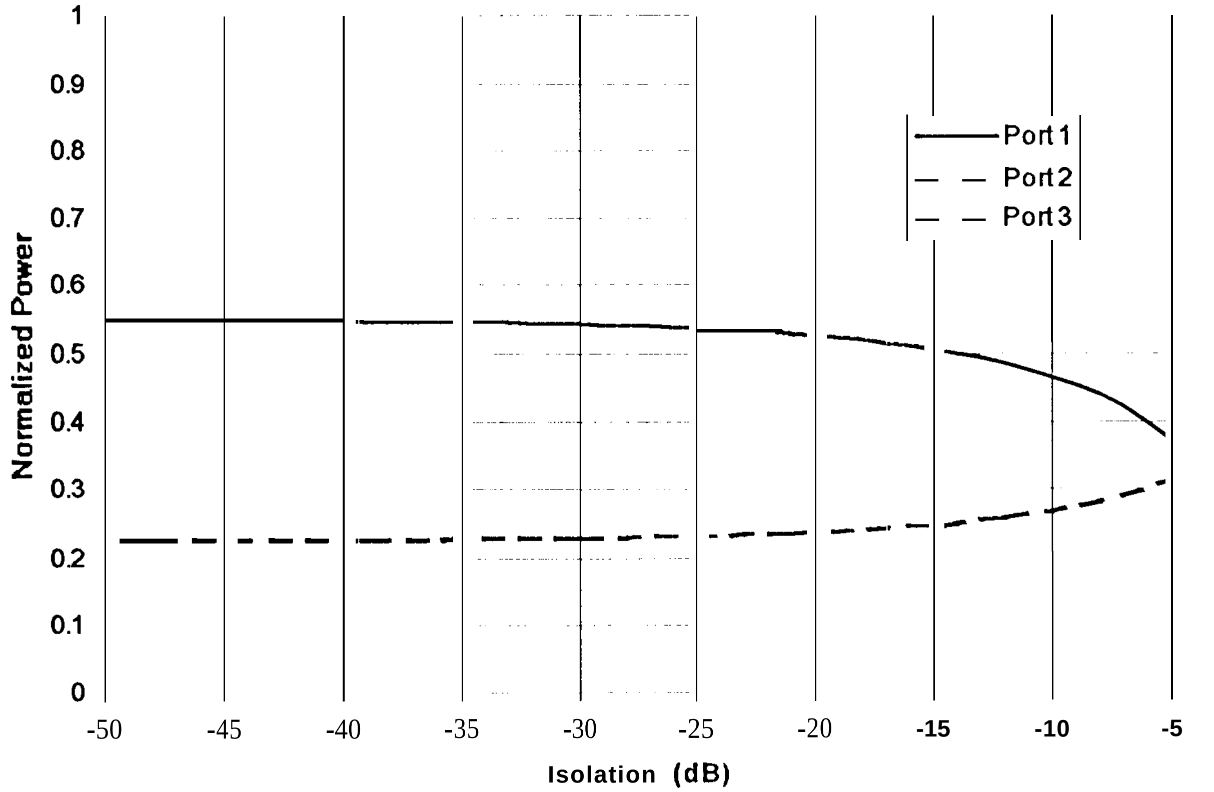


Figure 7 The power, in each of the ports, for zero dissipation.

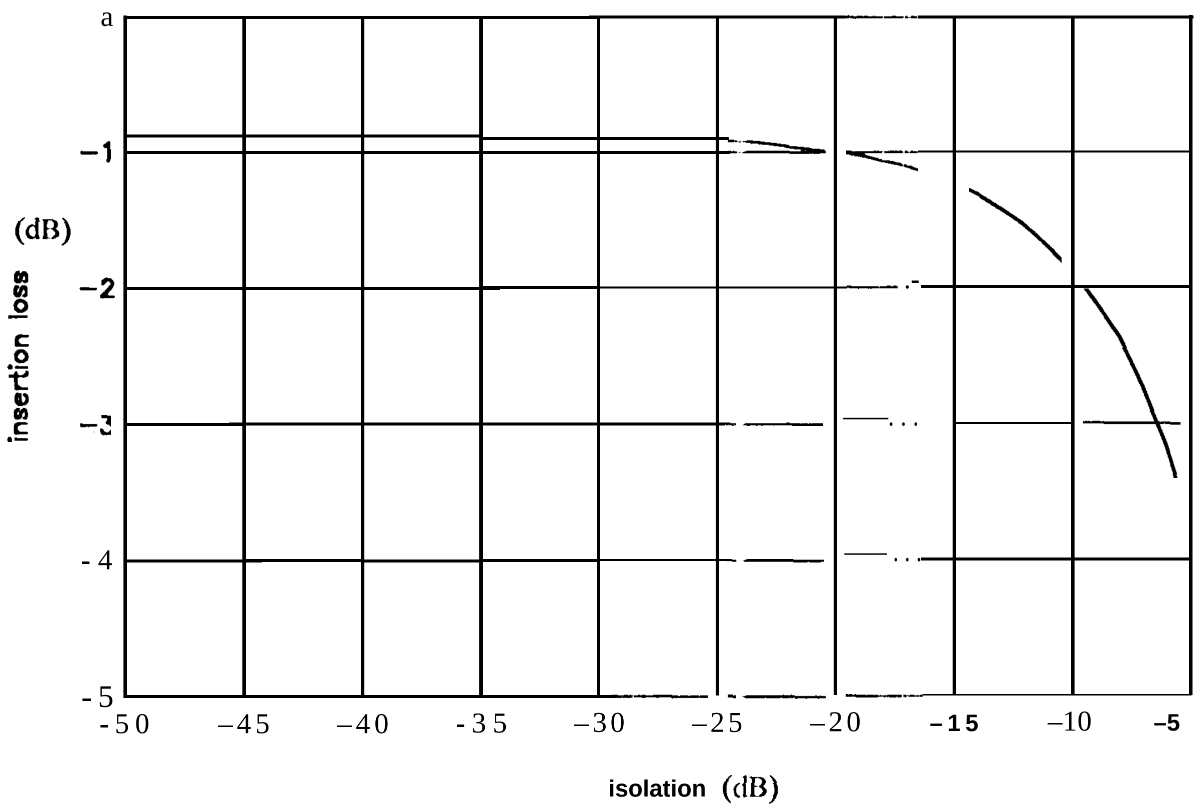


Figure 8 Trade-off between insertion loss & isolation.

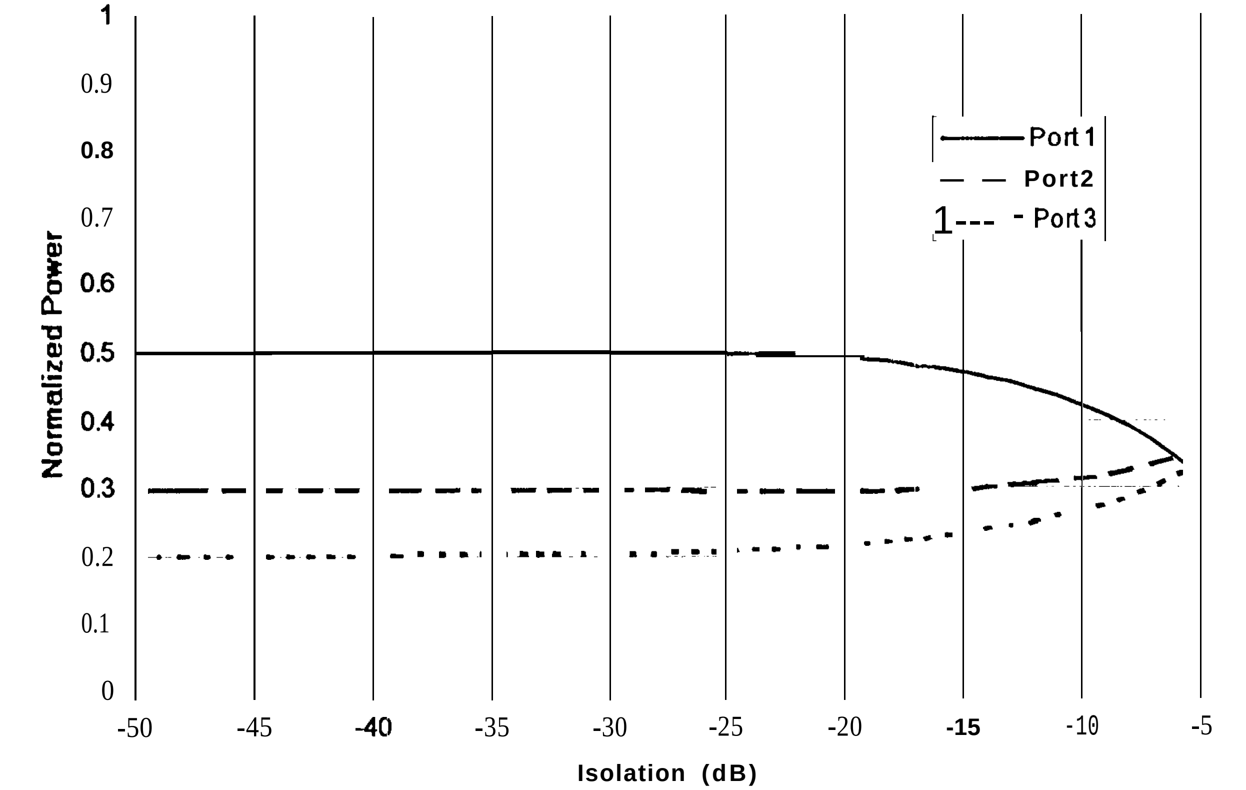


Figure 9 The power, in each of the ports, for zero dissipation.